

A nonlinear prerelativistic approach to mathematical representation of vacuum electromagnetism.

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Abstract

This paper presents an alternative pre-relativistic approach to the vacuum case of classical electrodynamics represented by vacuum Maxwell equations. Our view is based on the understanding that the corresponding differential equations should be dynamical in nature and the physical relations represented by them should be directly verifiable at least in principle, so they must represent local energy-momentum balance relations.

One must consider the source of nonlinearity in a field theory by starting with physically reasonable assumptions, ..., rather than merely generalizing the purely mathematical form of the lagrangian or field equations.

D.H.Delphenich

1 Introduction: General Notions about Physical Objects and Interactions

In correspondence with modern theoretical view on classical fields we consider time dependent and space propagating electromagnetic fields as flows of spatially finite physical entities which have been called in the early 20th century *photons*. Since the efforts made during the past years to find appropriate nonlinearizations of Maxwell vacuum equations [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15], and the seriously developed quantum theory have not resulted so far in appropriate, from our viewpoint, description of time stable entities of electromagnetic field nature with finite spatial support, in particular, adequate theoretical image of single photons as spatially finite physical objects with internal dynamical structure, explaining in such a way their time stability and spin nature, we decided to look back to the rudiments of the electromagnetic theory trying to reconsider its assumptions in order to come to equations giving appropriate solutions, in particular, solutions with spatially finite carrier at every moment of their existence and space-propagating as a whole, keeping, of course, their physical identity. This reconsideration brought us to the necessity to admit that introducing as clear as possible notions about the concepts *physical object* and *physical interaction* is strongly needed. Here we briefly present the view on these concepts that we shall follow throughout the paper.

When we speak about physical objects, e.g. classical particles, solid bodies, elementary particles, fields, etc., we always suppose that some *definite properties of the object under consideration do not*

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change during its time-evolution under the influence of the existing environment. The availability of such time-stable features of any physical object makes it *recognizable* among the other physical objects, on one hand, and guarantees its proper *identification* during its existence in time, on the other hand. Without such an availability of constant in time properties (features), which are due to the object's resistance and surviving abilities, we could hardly speak about objects and knowledge at all. So, for example, two classical mass particles together with the associated to them gravitational fields survive under the mutual influence of their gravitational fields through changing their states of motion: change of state compensates the consequences of the violated dynamical equilibrium that each of the two particles had been established with the physical environment before the two gravitational fields have begun perturbing each other.

The above view implies that three kinds of quantities will be necessary to describe as fully as possible the existence and the evolution of a given physical object:

1. **Proper (identifying) characteristics**, i.e. quantities which do NOT change during the entire existence of the object. The availability of such quantities allows to distinguish a physical object among the other ones.

2. **Kinematical characteristics**, i.e. quantities, which describe the allowed space-time evolution, where "allowed" means *consistent with the constancy of the identifying characteristics*.

3. **Dynamical characteristics**, i.e. quantities which are functions (explicit or implicit) of the *proper* and of the *kinematical* characteristics.

Some of the dynamical characteristics must have the following two important properties: they are in a definite degree **universal**, i.e. a class of physical objects (may be all physical objects) carry nonzero value of them (e.g. energy-momentum), and they are **conservative**, i.e. they may just be transferred from one physical object to another (in various forms) but no loss is allowed.

Hence, the evolution of a physical object subject to bearable/acceptable exterior influence (perturbation), coming from the existing environment, has three aspects:

1. *constancy of the proper (identifying) characteristics*,
2. *allowed kinematical evolution*,
3. *appropriate exchange of dynamical quantities with the physical environment guaranteeing object's time stability*.

Moreover, if the physical object under study is space-extended (continuous) and demonstrates internal structure and dynamics, i.e. available interaction of time-stable subsystems, it should be described by a many-component mathematical object, e.g. vector valued differential form, clearly, we must consider this *internal exchange* of some dynamical characteristics among the various subsystems of the object as essential feature determining in a definite extent object's integral appearance.

The above features suggest that the dynamical equations, describing locally the evolution of the object, may come from giving an explicit form of the quantities controlling the local internal and external exchange processes, i.e. writing down corresponding local balance equations. Hence, denoting the local quantities that describe the external exchange processes by $Q_i, i = 1, 2, \dots$, the object should be considered to be Q_i -free, $i = 1, 2, \dots$, if the corresponding integral values are time constant, which can be achieved only if Q_i obey differential equations presenting appropriately (implicitly or explicitly) corresponding local versions of the conservation laws (continuity equations). In case of absence of external exchange similar equations should describe corresponding *internal* exchange processes. The corresponding evolution in this latter case may be called *proper evolution*.

Summerizing, we may assume the rule that the available changes of a physical field system, or a recognizable subsystem of a given system, *must be referred somehow* to the very system, or to some of the subsystems, in order to evaluate their significance:

-if the referred quantity is zero, then the changes are admissible and the system/subsystem keeps its identity;

-if mutually referred quantities among subsystems establish dynamical equilibrium, i.e., each subsystem gains as much as it loses, then the whole system keeps its identity;

-if the referred quantity is not zero then the identity of the system is partially, or fully, lost, so, our system undergoes essential changes leading to becoming subsystem of another system, or to destruction, giving birth to new system(s).

In trying to formalize these views it seems appropriate to give some initial explicit formulations of some most basic features (properties) of what we call physical object, which features would lead us to a, more or less, adequate theoretical notion of our intuitive notion of a physical object. Anyway, the following properties of the theoretical concept "physical object" we consider as necessary:

1. It can be created during finite period(s) of time.
2. It can be destroyed during finite period(s) of time.
3. It occupies finite 3-volume at any moment of its existence, so it has spatial structure and may be considered as a system consisting of two or more interconnected subsystems.
4. It has a definite stability to withstand definite external disturbances.
5. It has definite conservation properties.
6. It necessarily carries sufficiently universal measurable quantities, e.g. energy-momentum.
7. It exists in an appropriate environment (called usually vacuum), which provides all necessary existence needs. Figuratively speaking, every stable physical object lives in a dynamical equilibrium with the outside world, which dynamical equilibrium may be realized in various regimes.
8. It can be detected by other physical objects through allowed exchanges of appropriate physical quantities, e.g. energy-momentum.
9. It may combine/coexist through interaction with other appropriate physical objects to form new objects/systems of higher level structure. In doing this it may keep its identity and can be recognized and identified throughout the existence of the system as its constituent/subsystem.
10. Its destruction gives necessarily birth to new objects, and this process respects definite rules of conservation. In particular, the available interaction energy among its subsystems may transform entirely, or partially, to kinetic one, and carried away by the newly created objects/systems.

The property *to be spatially finite* we consider as a very essential one. So, the above features do NOT allow the classical material points and the infinite classical fields (e.g. plane waves) to be considered as physical objects since the former have no structure and cannot be destroyed at all, and the latter carry infinite energy, so they cannot be finite-time-created. Hence, the Born-Infeld "principle of finiteness" [2] stating that "*a satisfactory theory should avoid letting physical quantities become infinite*" may be strengthened as follows:

All real physical objects are spatially finite entities and NO infinite values of the physical quantities carried by them are allowed.

Clearly, together with the purely qualitative features, physical objects carry important quantitatively described physical properties, and any external interaction may be considered as an exchange of the corresponding quantities provided both the object and the corresponding environment carry them. Hence, the more universal is a physical quantity the more useful for us it is, and this moment determines the exclusively important role of energy-momentum, which modern physics considers as the most universal one, i.e. it is more or less assumed that:

All physical objects necessarily carry energy-momentum and most of them are able in a definite extent to lose and gain energy-momentum.

The above notes more or less say that we make use of the term "physical object" when we consider it from integral point of view, i.e. when its stability against external perturbations is guaranteed. We make use of the term "physical system" when time-stable interacting subsystems are possible to be recognized/identified and the behavior of the system as a whole, i.e. considered from outside, we try to consider as seriously dependent on its internal dynamical structure, i.e. on an available stable interaction of its time-recognizable/time-identifiable subsystems. Therefore we shall follow the rule:

Physical recognizability of time-stable subsystems of a physical system requires corresponding mathematical recognizability in the theory.

We note that further we approach the problem from *nonrelativistic* viewpoint.

Since any physical interaction presupposes equilibrium or nonequilibrium dynamical exchange, i.e. flows of some physical quantities among the subsystems, the above notes make us turn to the intrinsic dynamical nature of vector fields (on \mathbb{R}^3) as formal local representatives of real flows, or real dynamical equilibrium.

2 Maxwell stress tensors generated by one and two vector fields

From formal point of view the most natural mathematical object to be considered as stress generating, seems to be any vector field, because every vector field defined on an arbitrary manifold M generates 1-parameter family φ_t of (local in general) diffeomorphisms of M . Therefore, having defined a vector field X on M we can consider for each $t \in \mathbb{R}$ the corresponding diffeomorphic image $\varphi_t(U)$ of any region $U \subset M$. Hence, if the spatially finite object under consideration occupies for each t the finite region $U_t \subset \mathbb{R}^3$, carries dynamical structure and propagates in the 3-space, should stay identical to itself during propagation, the available 1-parameter group of diffeomorphisms can be considered as the appropriate mathematical tool to describe its propagation as a whole, as well as, from local point of view, since every admissible change described by φ_t has unique opposite, described by φ_{-t} , i.e. nothing essential has been lost irreversibly and the object stays identical to itself.

Let now X be a vector field on the euclidean space (\mathbb{R}^3, g) , where g is the euclidean metric in $T\mathbb{R}^3$ having in the canonical global coordinates $(x^1, x^2, x^3 = x, y, z)$ components $g_{11} = g_{22} = g_{33} = 1$ and $g_{12} = g_{13} = g_{23} = 0$. The induced euclidean metric in $T^*\mathbb{R}^3$ has in the dual bases the same components and will be denoted further by the same letter g . The corresponding isomorphisms between the tangent and cotangent spaces and their tensor, exterior and symmetric products will be denoted by the same symbol \tilde{g} , so (summation on the repeating indecies is assumed)

$$\tilde{g}\left(\frac{\partial}{\partial x^i}\right) = g^{ik}\left(\frac{\partial}{\partial x^k}\right) = dx^i, \quad (\tilde{g})^{-1}(dx^i) = \frac{\partial}{\partial x^i} \quad \dots$$

Having another vector field Y on \mathbb{R}^3 we can form their scalar product

$$g(X, Y) \equiv X \cdot Y = g_{ij} X^i Y^j = X_i Y^i = X_1 Y^1 + X_2 Y^2 + X_3 Y^3.$$

According to classical vector analysis on \mathbb{R}^3 [16] for the differential of the function $g(X, Y)$ we can write

$$dg(X, Y) = (X \cdot \nabla)Y + (Y \cdot \nabla)X + X \times \text{rot}(Y) + Y \times \text{rot}(X),$$

where in our coordinates

$$X \cdot \nabla = \nabla_X = X^i \frac{\partial}{\partial x^i}, \quad (X \cdot \nabla)Y = \nabla_X Y = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j},$$

" \times " is the usual vector product, and

$$(\text{rot} X)^i = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}, \frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}, \frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2} \right).$$

The Hodge $*_g$ -operator acts in these coordinates as follows:

$$\begin{aligned} *dx &= dy \wedge dz, \quad *dy = -dx \wedge dz, \quad *dz = dx \wedge dy, \\ *(dx \wedge dy) &= dz, \quad *(dx \wedge dz) = -dy, \quad *(dy \wedge dz) = dx, \\ *(dx \wedge dy \wedge dz) &= 1, \quad *1 = dx \wedge dy \wedge dz. \end{aligned}$$

Corollary. The following relation holds (\mathbf{d} denotes the exterior derivative):

$$\text{rot} X = (\tilde{g})^{-1} * \mathbf{d} \tilde{g}(X), \quad \text{or} \quad \tilde{g}(\text{rot} X) = * \mathbf{d} \tilde{g}(X).$$

Assume now that in the above expression for $\mathbf{d}g(X, Y)$ we put $X = Y$. We obtain

$$\frac{1}{2} \mathbf{d}g(X, X) = \frac{1}{2} \mathbf{d}(X^2) = X \times \text{rot} X + (X \cdot \nabla) X = X \times \text{rot} X + \nabla_X X.$$

In components, the last term on the right reads

$$(\nabla_X X)^j = X^i \nabla_i X^j = \nabla_i (X^i X^j) - X^j \nabla_i X^i = \nabla_i (X^i X^j) - X^j \text{div} X,$$

where (denoting by L_X the Lie derivative along X)

$$\text{div} X = *L_X (dx \wedge dy \wedge dz) = * \left(\frac{\partial X^i}{\partial x^i} dx \wedge dy \wedge dz \right) = \frac{\partial X^i}{\partial x^i}.$$

Substituting into the preceding relation, replacing $\mathbf{d}(X^2)$ by $(\nabla_i \delta_j^i X^2) dx^j$ and making some elementary transformations we obtain

$$\nabla_i \left(X^i X^j - \frac{1}{2} g^{ij} X^2 \right) = [(\text{rot} X) \times X + X \text{div} X]^j.$$

The symmetric 2-tensor

$$M^{ij} = X^i X^j - \frac{1}{2} g^{ij} X^2$$

we shall call further Maxwell stress tensor generated by the (arbitrary) vector field $X \in \mathfrak{X}(\mathbb{R}^3)$. Clearly, when we raise and lower indices in canonical coordinates with \tilde{g} we shall have the following component relations:

$$M_{ij} = M_i^j = M^{ij}$$

which does not mean, of course, that we equalize quantities being elements of different linear spaces.

We note now the easily verified relation between the vector product " \times " and the wedge product in the space of 1-forms on \mathbb{R}^3 :

$$\begin{aligned} X \times Y &= (\tilde{g})^{-1} (*(\tilde{g}(X) \wedge \tilde{g}(Y))) \\ &= (\tilde{g})^{-1} \circ i(X \wedge Y)(dx \wedge dy \wedge dz), \quad X, Y \in \mathfrak{X}(\mathbb{R}^3). \end{aligned}$$

We shall prove now the following

Proposition. If $\alpha = \tilde{g}(X)$ then the following relation holds ($i(X)\mathbf{d}\alpha$ means $X^i\mathbf{d}\alpha_{ij}dx^j$):

$$\tilde{g}(\text{rot } X \times X) = i(X)\mathbf{d}\alpha = - * (\alpha \wedge *\mathbf{d}\alpha).$$

Proof.

$$\tilde{g}(\text{rot } X \times X) = \tilde{g} \circ (\tilde{g})^{-1} * (\tilde{g}(\text{rot } X) \wedge \tilde{g}(X)) = - * (\alpha \wedge *\mathbf{d}\alpha).$$

For the component of $i(X)\mathbf{d}\alpha$ before dx we obtain

$$-X^2 \left(\frac{\partial \alpha_2}{\partial x^1} - \frac{\partial \alpha_1}{\partial x^2} \right) - X^3 \left(\frac{\partial \alpha_3}{\partial x^1} - \frac{\partial \alpha_1}{\partial x^3} \right),$$

and the same quantity is easily obtained for the component of $[- * (\alpha \wedge *\mathbf{d}\alpha)]$ before dx . The same is true for the components of the two 1-forms before dy and dz . The proposition is proved.

Hence, $\mathbf{d}\alpha = \mathbf{d}\tilde{g}(X)$ is the 2-form across which the vector field X will drag the points of the finite region $U \subset \mathbb{R}^3$, and local interaction between X and $\mathbf{d}\alpha$ will take place if $i(X)\mathbf{d}\alpha$ is not zero.

As for the second term $X \text{div} X$ of the divergence $\nabla_j M_j^i dx^i$, since $\mathbf{d} * \alpha = \text{div} X (dx \wedge dy \wedge dz)$, we easily obtain

$$i(\tilde{g}^{-1}(*\alpha))\mathbf{d} * \alpha = X \text{div} X.$$

Hence, additionally, the 2-vector $\tilde{g}^{-1}(*\alpha)$ will drag the points of $U \subset \mathbb{R}^3$ across the 3-form $\mathbf{d} * \alpha$.

We can write now

$$\nabla_i M_j^i dx^j = [i(X)\mathbf{d}\alpha + i(\tilde{g}^{-1}(*\alpha))\mathbf{d} * \alpha]_j dx^j.$$

One formal suggestion that comes from this relation is that the interior product of a multivector and a differential form is appropriate quantity to be used as a quantitative measure of physical interaction.

Another suggestion is that this relation may be understood in the sense, that our physical field, initially represented by the vector field X , can be equally well represented by two objects: $\alpha = \tilde{g}(X)$ and $*\alpha$. Moreover,

- there is no available local nonzero interaction stress between α and $*\alpha$,
- each of them interacts with itself only, so that there is no stress exchange between α and $*\alpha$.

The naturally isolated two terms in $\nabla_i M_j^i dx^j$ suggest: any realizable dynamical process represented by the vector field X to be described by the two objects $(\alpha, *\alpha)$, and the time-recognizable nature of α and $*\alpha$ to be guaranteed by the following equations:

$$i(X)\mathbf{d}\alpha = 0, \quad i(\tilde{g}^{-1}(*\alpha))\mathbf{d} * \alpha = 0, \quad \text{i.e.} \quad X \times \text{rot} X = 0, \quad \text{div} X = 0.$$

In the above consideration the role of the euclidean metric g was somehow overlooked since no derivatives of g appeared explicitly. We are going now to come to these equations paying due respect to g .

We shall work further with the 1-form $\alpha = \tilde{g}(X)$, where the vector field X generates stress according to the Maxwell stress tensor $M_j^i(X)$. We want to see how the 3-form $\alpha \wedge *\alpha = i(X)\alpha \cdot dx \wedge dy \wedge dz = i(X)\alpha \cdot \omega$ changes along an arbitrary vector field $Y \neq X$, so we have to find the corresponding Lie derivative.

$$\begin{aligned} L_Y(\alpha \wedge *\alpha) &= (L_Y \alpha) \wedge *\alpha + \alpha \wedge L_Y(*\alpha) \\ &= (L_Y \alpha) \wedge *\alpha + \alpha \wedge *L_Y \alpha + \alpha \wedge [L_Y, *_1]\alpha = 2(L_Y \alpha) \wedge *\alpha + \alpha \wedge [L_Y, *_1]\alpha, \end{aligned}$$

where $[L_Y, *]_1$ is the commutator $L_Y \circ * - * \circ L_Y$, and the index of $*$ denotes the degree of the form it is applied to. Further we get

$$(L_Y \alpha) \wedge * \alpha = (i_Y \mathbf{d} \alpha) \wedge * \alpha - \langle \alpha, Y \rangle \cdot \mathbf{d} * \alpha + \mathbf{d} \langle \alpha, Y \rangle \cdot * \alpha.$$

Noting that $L_Y(\alpha \wedge * \alpha) = \mathbf{d}(\alpha^2 i_Y \omega)$ and the relation between the exterior derivative \mathbf{d} and the coderivative δ in euclidean case for p-forms, given by

$$(-1)^{p(n-p)} \delta_p \circ *_{n-p} = *_{(n-p+1)} \circ \mathbf{d}_{(n-p)}$$

we obtain consecutively

$$\begin{aligned} \mathbf{d} \left(\frac{1}{2} \alpha_i \alpha^i i_Y \omega - \langle \alpha, Y \rangle * \alpha \right) &= -[\alpha^i (\mathbf{d} \alpha)_{ij} + \alpha_j \operatorname{div} X] Y^j \omega + \frac{1}{2} \alpha \wedge [L_Y, *]_1 \alpha, \\ \delta \circ *_2 \left(\frac{1}{2} \alpha_i \alpha^i i_Y \omega - \langle \alpha, Y \rangle * \alpha \right) &= -[\alpha^i (\mathbf{d} \alpha)_{ij} + \alpha_j \operatorname{div} X] Y^j + \frac{1}{2} \alpha \wedge [L_Y, *]_1 \alpha, \\ *_2 \langle \alpha, Y \rangle * \alpha &= \langle \alpha, Y \rangle \cdot \alpha, \\ \delta \left(\frac{1}{2} \alpha_i \alpha^i g_{jk} - \alpha_j \alpha_k \right) Y^j dx^k &= -\nabla_j \left[\left(\frac{1}{2} \alpha_i \alpha^i \delta_k^j - \alpha_k \alpha^j \right) Y^k \right]. \end{aligned}$$

So we get

$$\nabla_j \left[\left(\frac{1}{2} \alpha_i \alpha^i \delta_k^j - \alpha_k \alpha^j \right) Y^k \right] = [\alpha^i (\mathbf{d} \alpha)_{ij} + \alpha_j \operatorname{div} X] Y^j + \frac{1}{2} \alpha \wedge [L_Y, *]_1 \alpha.$$

Hence, if $[L_Y, *]_1 = 0$, for example when Y is a Killing vector field for g , and the equations for our 1-form $\alpha = \tilde{g}(X)$ are

$$\alpha^i (\mathbf{d} \alpha)_{ij} = 0, \quad \alpha_j \operatorname{div} X = 0, \quad i, j = 1, 2, 3$$

then the co-closed 1-form $M_{ij} Y^i dx^j$ defines the closed 2-form $*(M_{ij} Y^i dx^j)$, so, its integral over a closed 2-surface that separates the 3-volume where, by supposition, α is different from zero, gives a conservative quantity and the nature of this conservative quantity is connected with the nature of the vector field Y , which is a Killing vector in our 3-dimensional case. Of course, the term "conservative" here is, more or less, trivial, since just stress is available, no space propagation takes place.

Finally we note that if the vector fields X_1, X_2, \dots, X_p describe time-stable physical exchanges, we can form the sum of their Maxwell stress tensors

$$M^{ij}(X_1) + M^{ij}(X_2) + \dots + M^{ij}(X_p) = \sum_{k=1}^p \left[(X_k)^i (X_k)^j - \frac{1}{2} (X_k)^2 g^{ij} \right],$$

which is not equal to the Maxwell stress tensor of their sum $M^{ij}(X_1 + \dots + X_p)$.

As for solutions of the above equations, we may suppose two classes of solutions:

1. Linear, i.e. those satisfying $\mathbf{d} \alpha = 0, \mathbf{d} * \alpha = 0$. These are generated by a function f satisfying the Laplace equation $\Delta f = 0$ with $\alpha = \mathbf{d} f$.

2. Those, satisfying $\mathbf{d} \alpha \neq 0$ but $\alpha^i \mathbf{d} \alpha_{ij} = 0$ and $\mathbf{d} * \alpha = 0$. Note that for such solutions the determinant $\det |(\mathbf{d} \alpha)_{ij}|, i, j = 1, 2, 3$, is always zero, so the homogenous linear system

$$\alpha^i(x, y, z) \mathbf{d} \alpha_{ij}(x, y, z) = 0$$

allows to express explicitly X through the derivatives of its components.

Of course, all these solutions are static, so they can not serve as models of spatially propagating finite physical objects with dynamical structure.

We pass now to the case of *two* vector fields.

Let V and W be two vector fields on our euclidean 3-space. Summing up the corresponding two Maxwell stress tensors we obtain the identity:

$$\begin{aligned}\nabla_i M_{(V,W)}^{ij} &\equiv \nabla_i \left(V^i V^j + W^i W^j - \delta^{ij} \frac{V^2 + W^2}{2} \right) = \\ &= [(\text{rot } V) \times V + V \text{div } V + (\text{rot } W) \times W + W \text{div } W]^j.\end{aligned}$$

Let now $(a(x, y, z), b(x, y, z))$ be two arbitrary functions on \mathbb{R}^3 . We consider the transformation

$$(V, W) \rightarrow (V a - W b, V b + W a).$$

Corollary.

The tensor $M_{(V,W)}$ transforms to $(a^2 + b^2)M_{(V,W)}$.

Corollary.

The transformations $(V, W) \rightarrow (V a - W b, V b + W a)$ do not change the eigen directions structure of $M_{(V,W)}^{ij}$.

Corollary.

If $a = \cos \alpha, b = \sin \alpha$, where $\alpha = \alpha(x, y, z)$ then the tensor $M_{(V,W)}$ stays invariant:

$$M_{(V,W)} = M_{(V \cos \alpha - W \sin \alpha, V \sin \alpha + W \cos \alpha)}.$$

The expression inside the parentheses above, denoted by M^{ij} , looks formally the same as the introduced by Maxwell tensor $M^{ij}(\mathbf{E}, \mathbf{B})$ from physical considerations concerned with the electromagnetic stress energy properties of continuous media in presence of external electromagnetic field (\mathbf{E}, \mathbf{B}) . Hence, any vector V , or any couple of vectors (V, W) , defines such tensor which we denote by M_V , or $M_{(V,W)}$, and call **Maxwell stress tensor**. The term, "stress" in this general mathematical setting could be justified in the following way. Every vector field on \mathbb{R}^3 generates corresponding flow by means of the trajectories started from some domain $U_{t=0} \subset \mathbb{R}^3$, where t is an arbitrary parameter: at $t > 0$ the domain $U_{t=0}$ is diffeomorphically transformed to a new domain $U_t \subset \mathbb{R}^3$. Having two appropriate vector fields on \mathbb{R}^3 we may obtain two *compatible* flows, i.e. to expect the points of an interesting for us domain $U_{t=0} \subset \mathbb{R}^3$ to be forced to accordingly move to new positions inside a new appropriate domain.

We emphasize the following moments:

1. The identity satisfied by $M_{(V,W)}$ is purely mathematical, and t in U_t is *arbitrary* parameter, not time in general;
2. On the two sides of this identity stay well defined coordinate free quantities;
3. The tensors $M_{(V,W)}$ do not introduce interaction stress: the full stress is the sum of the stresses generated by each one of the couple (V, W) .

Physically, we may say that the corresponding physical medium that occupies the spatial region U_o and is parametrized by the points of the mathematical subregion $U_o \subset \mathbb{R}^3$, is subject to *compatible* and *admissible* physical "stresses" generated by physical interactions mathematically described by the vector fields (V, W) , and these physical stresses are quantitatively described by

the corresponding physical interpretation of the tensor $M_{(V,W)}$. Clearly, we could extend the couple (V, W) to more vectors (V_1, V_2, \dots, V_p) , but then the mentioned invariance properties of $M_{(V,W)}$ may be lost, or appropriately extended.

We note that the stress tensor M^{ij} appears as been subject to the divergence operator, and if we interpret the components of M^{ij} as physical stresses, then its divergence acquires, in general, the physical interpretation of force density. Of course, in the static situation as it is given by the relation considered, no stress propagation is possible, so at every point the local forces mutually compensate: $\nabla_i M^{ij} = 0$. In electrodynamics, if propagation is allowed, then the force field may NOT be zero: $\nabla_i M^{ij} \neq 0$, and we may identify it as a **real time-change** of appropriately defined momentum density $\mathbf{P}_{(\mathbf{E}, \mathbf{B})}$. So, assuming some expression for this momentum density $\mathbf{P}_{(\mathbf{E}, \mathbf{B})}$ we are ready to write down corresponding dynamical field equation through equalizing the spatially directed force densities $\nabla_i M^{ij}$ with the momentum density changes along the time variable, i.e., equalizing $\nabla_i M^{ij}$ with the (ct) -derivative of $\mathbf{P}_{(\mathbf{E}, \mathbf{B})}$, where $c = \text{const}$ is the translational propagation velocity of the momentum density flow of the physical system considered. In order to find how to choose $\mathbf{P}_{(\mathbf{E}, \mathbf{B})}$ in case of *free* EM-field we have to turn to the intrinsic *physical* properties of the field. As the past years showed, namely $M_{(\mathbf{E}, \mathbf{B})}^{ij}$ is the appropriate carrier of the physical properties of the field, so, it seems natural to turn to the eigen properties of $M_{(\mathbf{E}, \mathbf{B})}^{ij}$.

3 Eigen properties of Maxwell stress tensor

We consider $M^{ij}(\mathbf{E}, \mathbf{B})$ at some point $p \in \mathbb{R}^3$ and assume that in general the vector fields \mathbf{E} and \mathbf{B} are lineary independent, so $\mathbf{E} \times \mathbf{B} \neq 0$. Let the coordinate system be chosen such that the coordinate plane (x, y) to coincide with the plane defined by $\mathbf{E}(p), \mathbf{B}(p)$. In this coordinate system $\mathbf{E} = (E_1, E_2, 0)$ and $\mathbf{B} = (B_1, B_2, 0)$, so, identifying the contravariant and covariant indices through the Euclidean metric δ^{ij} (so that $M^{ij} = M_j^i = M_{ij}$), we obtain the following nonzero components of the stress tensor:

$$M_1^1 = (E^1)^2 + (B^1)^2 - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2); \quad M_2^1 = M_1^2 = E^1 E_2 + B_1 B^2;$$

$$M_2^2 = (E^2)^2 + (B^2)^2 - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2); \quad M_3^3 = -\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2).$$

Since $M_1^1 = -M_2^2$, the trace of M is $Tr(M) = -\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$.

The eigen value equation acquires the simple form

$$[(M_1^1)^2 - (\lambda)^2] + (M_2^1)^2 (M_3^3 - \lambda) = 0.$$

The corresponding eigen values are

$$\lambda_1 = -\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2); \quad \lambda_{2,3} = \pm \sqrt{(M_1^1)^2 + (M_2^1)^2} = \pm \frac{1}{2} \sqrt{(I_1)^2 + (I_2)^2},$$

where $I_1 = \mathbf{B}^2 - \mathbf{E}^2$, $I_2 = 2\mathbf{E} \cdot \mathbf{B}$.

The corresponding to λ_1 eigen vector Z_1 must satisfy the equation

$$\mathbf{E}(\mathbf{E} \cdot Z_1) + \mathbf{B}(\mathbf{B} \cdot Z_1) = 0,$$

and since (\mathbf{E}, \mathbf{B}) are lineary independent, the two coefficients $(\mathbf{E} \cdot Z_1)$ and $(\mathbf{B} \cdot Z_1)$ must be equal to zero, therefore, $Z_1 \neq 0$ must be orthogonal to \mathbf{E} and \mathbf{B} , i.e. Z_1 must be colinear to $\mathbf{E} \times \mathbf{B}$:

The other two eigen vectors $Z_{2,3}$ satisfy correspondingly the equations

$$\mathbf{E}(\mathbf{E} \cdot Z_{2,3}) + \mathbf{B}(\mathbf{B} \cdot Z_{2,3}) = \left[\pm \frac{1}{2} \sqrt{(I_1)^2 + (I_2)^2} + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \right] Z_{2,3}. \quad (*)$$

Taking into account the easily verified relation

$$\frac{1}{4}[(I_1)^2 + (I_2)^2] = \left(\frac{\mathbf{E}^2 + \mathbf{B}^2}{2} \right)^2 - |\mathbf{E} \times \mathbf{B}|^2,$$

so that

$$\frac{\mathbf{E}^2 + \mathbf{B}^2}{2} - |\mathbf{E} \times \mathbf{B}| \geq 0,$$

we conclude that the coefficient before $Z_{2,3}$ on the right is always different from zero, therefore, the eigen vectors $Z_{2,3}(p)$ lie in the plane defined by $(\mathbf{E}(p), \mathbf{B}(p))$, $p \in \mathbb{R}^3$. In particular, the above mentioned transformation properties of the Maxwell stress tensor $M(V, W) \rightarrow (a^2 + b^2)M(V, W)$ show that the corresponding eigen directions do not change under the transformation $(V, W) \rightarrow (V a - W b, V b + W a)$.

The above consideration suggests: *the intrinsically allowed dynamical abilities of the field are: translational along $(\mathbf{E} \times \mathbf{B})$, and rotational inside the plane defined by (\mathbf{E}, \mathbf{B}) , hence, we may expect finding field objects the propagation of which shows intrinsic local consistency between rotation and translation.*

It is natural to ask now under what conditions the very \mathbf{E} and \mathbf{B} may be eigen vectors of $M(\mathbf{E}, \mathbf{B})$? Assuming $\lambda_2 = \frac{1}{2} \sqrt{(I_1)^2 + (I_2)^2}$ and $Z_2 = \mathbf{E}$ in the above relation (*) and having in view that $\mathbf{E} \times \mathbf{B} \neq 0$ we obtain that $\mathbf{E}(\mathbf{E}^2) + \mathbf{B}(\mathbf{E} \cdot \mathbf{B})$ must be proportional to \mathbf{E} , so, $\mathbf{E} \cdot \mathbf{B} = 0$, i.e. $I_2 = 0$. Moreover, substituting now $I_2 = 0$ in that same relation we obtain

$$\mathbf{E}^2 = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) = \mathbf{B}^2, \quad \text{i.e., } I_1 = 0.$$

The case "-" sign before the square root, i.e. $\lambda_3 = -\frac{1}{2} \sqrt{(I_1)^2 + (I_2)^2}$, leads to analogical conclusions just the role of \mathbf{E} and \mathbf{B} is exchanged.

Corollary. \mathbf{E} and \mathbf{B} may be eigen vectors of $M(\mathbf{E}, \mathbf{B})$ only if $I_1 = I_2 = 0$.

The above notices suggest to consider in a more detail the case $\lambda_2 = -\lambda_3 = 0$ for the vacuum case. We shall show, making use of the Lorentz transformation in 3-dimensional form that, if these two relations do not hold then under $\mathbf{E} \times \mathbf{B} \neq 0$ the translational velocity of propagation is less then the speed of light in vacuum c . Recall first the transformation laws of the electric and magnetic vectors under Lorentz transformation defined by the 3-velocity vector \mathbf{v} and corresponding parameter $\beta = v/c$, $v = |\mathbf{v}|$. If γ denotes the factor $1/\sqrt{1 - \beta^2}$ then we have

$$\begin{aligned} \mathbf{E}' &= \gamma \mathbf{E} + \frac{1 - \gamma}{v^2} \mathbf{v}(\mathbf{E} \cdot \mathbf{v}) + \frac{\gamma}{c} \mathbf{v} \times \mathbf{B}, \\ \mathbf{B}' &= \gamma \mathbf{B} + \frac{1 - \gamma}{v^2} \mathbf{v}(\mathbf{B} \cdot \mathbf{v}) - \frac{\gamma}{c} \mathbf{v} \times \mathbf{E}. \end{aligned}$$

Assume first that $I_2 = 2\mathbf{E} \cdot \mathbf{B} = 0$, i.e. \mathbf{E} and \mathbf{B} are orthogonal, so, in general, in some coordinate system we shall have $\mathbf{E} \times \mathbf{B} \neq 0$.

If $I_1 > 0$, i.e. $|\mathbf{E}| < |\mathbf{B}|$, we shall show that the conditions $\mathbf{E}' = 0, \mathbf{v} \cdot \mathbf{B} = 0, \infty > \gamma > 0$ are compatible. In fact, these assumptions lead to $\gamma \mathbf{v} \cdot \mathbf{E} + (1 - \gamma)(\mathbf{E} \cdot \mathbf{v}) = 0$, i.e. $\mathbf{E} \cdot \mathbf{v} = 0$.

Thus, $c|\mathbf{E}| = v|\mathbf{B}||\sin(\mathbf{v}, \mathbf{B})|$, and since $\mathbf{v} \cdot \mathbf{B} = 0$ then $|\sin(\mathbf{v}, \mathbf{B})| = 1$. It follows that the speed $v = c\frac{|\mathbf{E}|}{|\mathbf{B}|} < c$ is allowed.

If $I_1 < 0$, i.e. $|\mathbf{E}| > |\mathbf{B}|$, then the conditions $\mathbf{B}' = 0$ and $\mathbf{v} \cdot \mathbf{E} = 0$ analogically lead to the conclusion that the speed $v = c\frac{|\mathbf{B}|}{|\mathbf{E}|} < c$ is allowed.

Assume now that $I_2 = 2\mathbf{E} \cdot \mathbf{B} \neq 0$. We are looking for a reference frame K' such that $\mathbf{E}' \times \mathbf{B}' = 0$, while in the reference frame K we have $\mathbf{E} \times \mathbf{B} \neq 0$. We choose the relative velocity \mathbf{v} such that $\mathbf{v} \cdot \mathbf{E} = \mathbf{v} \cdot \mathbf{B} = 0$. Under these conditions the equation $\mathbf{E}' \times \mathbf{B}' = 0$ reduces to

$$\mathbf{E} \times \mathbf{B} + \frac{\mathbf{v}}{c}(\mathbf{E}^2 + \mathbf{B}^2) = 0, \quad \text{so,} \quad \frac{v}{c} = |\mathbf{E} \times \mathbf{B}|/(\mathbf{E}^2 + \mathbf{B}^2).$$

Now, from the above mentioned inequality $\mathbf{E}^2 + \mathbf{B}^2 - 2|\mathbf{E} \times \mathbf{B}| \geq 0$ it follows that $\frac{v}{c} < 1$.

Physically, these considerations show that under nonzero I_1 and I_2 the translational velocity of propagation of the field, and of the stress field energy density of course, can be eliminated by passing to appropriate frame of mass bodies. Hence, the only realistic choice for the vacuum case, where this velocity is assumed by definition to be equal to the vacuum light speed c and can not be eliminated by passing to appropriate physical frame, is $I_1 = I_2 = 0$, which is equivalent to $\mathbf{E}^2 + \mathbf{B}^2 = 2|\mathbf{E} \times \mathbf{B}|$. In view of this and assuming $|Tr(M)| = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ to be the stress energy density of the field, the names "electromagnetic energy flux" for the quantity $c\mathbf{E} \times \mathbf{B}$, and "momentum" for the quantity $\frac{1}{c}\mathbf{E} \times \mathbf{B}$, seem well justified without turning to any dynamical field equations.

These considerations suggest also that if $I_1 = 0$, i.e. $|\mathbf{E}|^2 = |\mathbf{B}|^2$ during propagation, then the energy density can be presented in terms of each of the two constituents, moreover, in this respect, both constituents have the same rights. Therefore, a local mutual energy exchange between the two subsystems formally represented in terms of \mathbf{E}, \mathbf{B} is not forbidden in general, but, if it takes place, it must be *simultaneous* and in *equal quantities*. Hence, under zero invariants $I_1 = 0$ and $I_2 = 2\mathbf{E} \cdot \mathbf{B} = 0$, internal energy redistribution between the two subsystems of the field would be allowed but such an exchange should occur *without available interaction energy* because the full energy density is always equal to the sum of the energy densities carried by the two subsystems of the field. But the above relation $\mathbf{E}^2 + \mathbf{B}^2 = 2|\mathbf{E} \times \mathbf{B}|$ and the required time-stability and propagation with velocity "c" of the field suggest/imply also *available internal momentum exchange to take place*, since this relation means that the energy density is always strongly proportional to the momentum density magnitude $\frac{1}{c}|\mathbf{E} \times \mathbf{B}|$. However, the important observation here is that, various combinations constructed out of the constituents \mathbf{E} and \mathbf{B} , e.g., $(\mathbf{E}\cos\alpha - \mathbf{B}\sin\alpha, \mathbf{E}\sin\alpha + \mathbf{B}\cos\alpha)$, where $\alpha(x, y, z, t)$ is a function, may be considered as possible representatives of the two recognizable subsystems since they carry the same energy $\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ and momentum $\frac{1}{c}|\mathbf{E} \times \mathbf{B}|$. Therefore, the assumption that the very \mathbf{E} and \mathbf{B} may be considered as appropriate mathematical images of recognizable time-stable subsystems of a time-dependent electromagnetic field does NOT seem adequate and has to be reconsidered.

Hence, which combinations of \mathbf{E} and \mathbf{B} deserve represent mathematically the two subsystems of a time-dependent and space-propagating electromagnetic (EM) field?

In view of these considerations we assume the following understanding:

Every real EM-field is built of two recognizable subsystems, the mathematical images of which are not the very (\mathbf{E}, \mathbf{B}) , but are expressed in terms of (\mathbf{E}, \mathbf{B}) , both these subsystems carry always the same quantity of energy-momentum, guaranteeing in this way that the supposed internal energy-momentum exchange will also be in equal quantities and simultaneous.

4 Real EM-fields viewed as built of two recognizable and permanently interacting subsystems.

In accordance with the above assumption the description of dynamical and space-propagating behavior of the field will need two appropriate mathematical objects to be constructed out of the two constituents (\mathbf{E}, \mathbf{B}) . These two mathematical objects must meet the required property that the two physical subsystems of the field carry always the same quantity of energy-momentum, and that any possible internal energy-momentum exchange between the two subsystems shall be *simultaneous and in equal quantities*.

We are going to consider time dependent fields, and begin with noting once again that the assumption that the energy density of the field coincides with $|Tr(M)| = \frac{1}{2}[\mathbf{E}^2 + \mathbf{B}^2]$ suggests that *there is NO interaction energy* between the two subsystems of the field: the full stress tensor (and the energy density, in particular) is a sum of the stress tensors carried separately the two subsystems. As we mentioned above, this does NOT mean that there is no energy exchange between the two subsystems of the field.

Now, following the above stated idea we have to find two appropriate mathematical images of the field which images are NOT represented directly by the electric \mathbf{E} and magnetic \mathbf{B} vectors, but are constructed out of them. In terms of these two appropriate mathematical representatives of the corresponding two partnering subsystems we must express the mentioned special kind of energy-momentum exchange, respecting in this way the fact that *neither* of the two constituents *is able to carry momentum separately*.

In view of the above we have to assume that the field keeps its identity through adopting some special and appropriate dynamical behavior according to its intrinsic capabilities. Since the corresponding dynamical/field behavior must be consistent with the properties of the intrinsic stress-energy-momentum nature of the field, we come to the conclusion that Maxwell stress tensor $M(\mathbf{E}, \mathbf{B})$ should play the basic role, and its zero-divergence in the static case should suggest how to determine the appropriate structure and allowed dynamics.

Recall that any member of the family

$$(\mathcal{E}, \mathcal{B}) = (\mathbf{E}, \mathbf{B}, \alpha) = (\mathbf{E} \cos \alpha - \mathbf{B} \sin \alpha; \mathbf{E} \sin \alpha + \mathbf{B} \cos \alpha), \quad \alpha = \alpha(x, y, z; t),$$

generates the same Maxwell stress tensor. So, the most natural assumption should read like this: **any member $(\mathbf{E}, \mathbf{B}, \alpha_1)$ of this α -family is looking for an energy-momentum exchanging partner $(\mathbf{E}, \mathbf{B}, \alpha_2)$ inside the family, and identifies itself through appropriate (local) interaction with the partner found, defining in this way corresponding dynamical behavior of the field.**

Simply speaking, a time-dependent EM-field is formally represented by two members of the above α -family, and the coupling $(\mathbf{E}, \mathbf{B}, \alpha_1) \leftrightarrow (\mathbf{E}, \mathbf{B}, \alpha_2)$ is unique.

Assuming for simplicity $\alpha_1 = 0$ then $(\mathcal{E}, \mathcal{B}, 0) \rightarrow (\mathbf{E}, \mathbf{B})$ and, making use of the local divergence of the Maxwell stress tensor and the time derivative of the local momentum flow of the field, we are going to find the value α_2 determining the corresponding partner-subsystem.

Further we shall call these two subsystems just partner-fields.

Recall the divergence

$$\begin{aligned} \nabla_i M^{ij} &\equiv \nabla_i \left(\mathbf{E}^i \mathbf{E}^j + \mathbf{B}^i \mathbf{B}^j - \delta^{ij} \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} \right) = \\ &= [(\text{rot } \mathbf{E}) \times \mathbf{E} + \mathbf{E} \text{div } \mathbf{E} + (\text{rot } \mathbf{B}) \times \mathbf{B} + \mathbf{B} \text{div } \mathbf{B}]^j. \end{aligned}$$

As we mentioned, in the static case, i.e. when the vector fields (\mathbf{E}, \mathbf{B}) do not depend on the time "coordinate" $\xi = ct$, NO propagation of field momentum density \mathbf{P} should take place, so, at every

point, where $(\mathbf{E}, \mathbf{B}) \neq 0$, the stress generated forces must mutually compensate, i.e. the divergence $\nabla_i M^{ij}$ should be equal to zero: $\nabla_i M^{ij} = 0$. In this static case Maxwell vacuum equations

$$\text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} = 0, \quad \text{rot } \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} = 0, \quad \text{div } \mathbf{E} = 0, \quad \text{div } \mathbf{B} = 0 \quad (\text{J.C.M.})$$

give: $\text{rot } \mathbf{E} = \text{rot } \mathbf{B} = 0$; $\text{div } \mathbf{E} = \text{div } \mathbf{B} = 0$, so, all static solutions to Maxwell equations determine a sufficient, but NOT necessary, condition that brings to zero the right hand side of the divergence through forcing each of the four vectors there to get zero values.

In the non-static case, i.e. when $\frac{\partial \mathbf{E}}{\partial t} \neq 0$; $\frac{\partial \mathbf{B}}{\partial t} \neq 0$, time change and propagation of field momentum density should take place, so, a full mutual compensation of the generated by the Maxwell stresses at every spatial point local forces may NOT be possible, which means $\nabla_i M^{ij} \neq 0$ in general. These local forces generate time-dependent momentum propagation inside the corresponding region. Therefore, if we want to describe this physical process of field energy-momentum density time change and spatial propagation we have to introduce explicitly the dependence of the local momentum \mathbf{P} on (\mathbf{E}, \mathbf{B}) , and to express the flow of the electromagnetic energy-momentum across an arbitrary static finite 2-dimensional surface S in two ways: in terms of $\nabla_i M^{ij} \neq 0$ and in terms of the time change of $\mathbf{P}(\mathbf{E}, \mathbf{B})$, and then to appropriately equilibrate them. Hence, we have to construct the corresponding two differential 2-forms to be integrated on S .

Note that compare to classical approach where the flows of the very \mathbf{E}, \mathbf{B} through some 2-surface are considered, we consider flows of quantities having direct stress-energy-momentum change sense.

In terms of $\mathfrak{F}^j = \nabla_i M^{ij} \neq 0$ the 2-form that is to be integrated on S is given by reducing $i_{\mathfrak{F}} \omega_o = i_{\mathfrak{F}}(\sqrt{|g|} dx^1 \wedge dx^2 \wedge dx^3) = *\tilde{g}(\mathfrak{F})$ on S , where $i_{\mathfrak{F}}$ denotes the interior product between the vector field \mathfrak{F} and the volume 3-form ω_o , i.e. the local flow of \mathfrak{F} across ω_o , and $*$ denotes the euclidean Hodge $*$. This local force flow $i_{\mathfrak{F}} \omega_o$ across S generates changes of the momentum density flow across S which should be equal to $\frac{d}{dt} \int_S *\tilde{g}(\mathbf{P}(\mathbf{E}, \mathbf{B}))$ (recall that t is considered as external parameter). We obtain (after restricting $*\tilde{g}(\mathbf{P}(\mathbf{E}, \mathbf{B}))$ and $*\tilde{g}(\mathfrak{F})$ on S):

$$\frac{d}{dt} \int_S *\tilde{g}(\mathbf{P}(\mathbf{E}, \mathbf{B})) = \int_S *\tilde{g}(\mathfrak{F}).$$

The explicit expression for $\mathbf{P}(\mathbf{E}, \mathbf{B})$, paying due respect to J.Poynting, and to J.J.Thomson, H.Poincare, M. Abraham, and in view of the huge, a century and a half available experience, has to be introduced by the following

Assumption: *The entire field momentum density is given by $\mathbf{P} := \frac{1}{c} \mathbf{E} \times \mathbf{B}$.*

According to this **Assumption**, to the above interpretation of the relation $\nabla_i M^{ij} \neq 0$, in view of the assumed by us local energy-momentum exchange approach to description of the dynamics of the field, in vector field terms and in canonical coordinates on \mathbb{R}^3 we come to the following vector differential equation (the 2-surface S is arbitrary and static)

$$\frac{\partial}{\partial \xi} (\mathbf{E} \times \mathbf{B}) = \mathfrak{F}, \quad \xi \equiv ct, \quad (*)$$

which is equivalent to

$$\left(\text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} + \mathbf{E} \text{div } \mathbf{E} + \left(\text{rot } \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \times \mathbf{B} + \mathbf{B} \text{div } \mathbf{B} = 0.$$

This last equation we write down in the following equivalent way:

$$\left(\text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} + \mathbf{B} \text{div } \mathbf{B} = - \left[\left(\text{rot } \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \times \mathbf{B} + \mathbf{E} \text{div } \mathbf{E} \right]. \quad (**)$$

The above relation (*) and its explicit forms we consider as mathematical adequate in energy-momentum-change terms of the electric-magnetic and magnetic-electric induction phenomena in the charge free case. We recall that it is usually assumed these induction phenomena to be described in classical electrodynamics by the following well known integral equations

$$\frac{d}{d\xi} \int_S * \tilde{g}(\mathbf{B})|_S = - \int_S * \tilde{g}(\text{rot } \mathbf{E})|_S \quad (\text{the Faraday induction law}),$$

$$\frac{d}{d\xi} \int_S * \tilde{g}(\mathbf{E})|_S = \int_S * \tilde{g}(\text{rot } \mathbf{B})|_S \quad (\text{the Maxwell displacement current law}),$$

where $(...)|_S$ means restriction of the corresponding 2-form to the 2-surface S .

We would like to note that these last Faraday-Maxwell relations have NO *direct* energy-momentum change-propagation (i.e. force flow) nature, so they could not be experimentally verified in a *direct* way. Our feeling is that, in fact, they are stronger than needed. So, on the corresponding solutions of these equations we'll be able to write down *formally adequate* energy-momentum change expressions, but the correspondence of these expressions with the experiment will crucially depend on the nature of these solutions. As is well known, the nature of the free solutions (with no boundary conditions) to Maxwell vacuum equations with spatially finite and smooth enough initial conditions requires strong time-instability (the Poisson theorem for the D'Alembert wave equation which each component of \mathbf{E} and \mathbf{B} must necessarily satisfy). And time-stability of time-dependent vacuum solutions usually requires spatial infinity (plane waves), which is physically senseless. Making calculations with spatially finite parts of these spatially infinite solutions may be practically acceptable, but from theoretical viewpoint assuming these equations for *basic* ones seems not acceptable since the relation "time stable physical object - exact free solution" is strongly violated.

Before to go further we write down the right hand side bracket expression in the following two equivalent ways:

$$\left[\left(\text{rot } \mathbf{B} + \frac{\partial(-\mathbf{E})}{\partial \xi} \right) \times \mathbf{B} + (-\mathbf{E}) \text{div } (-\mathbf{E}) \right];$$

$$\left[\left(\text{rot } (-\mathbf{B}) + \frac{\partial \mathbf{E}}{\partial \xi} \right) \times (-\mathbf{B}) + \mathbf{E} \text{div } \mathbf{E} \right].$$

These last two expressions can be considered as obtained from the left hand side of the above relation under the substitutions $(\mathbf{E}, \mathbf{B}) \rightarrow (\mathbf{B}, -\mathbf{E})$ and $(\mathbf{E}, \mathbf{B}) \rightarrow (-\mathbf{B}, \mathbf{E})$ respectively. Hence, the subsystem (\mathbf{E}, \mathbf{B}) chooses as a partner $(-\mathbf{B}, \mathbf{E})$, or $(\mathbf{B}, -\mathbf{E})$, so, $\alpha_2 = \pm \frac{\pi}{2}$. We conclude that the subsystem $(\mathbf{E}, \mathbf{B}, \alpha)$ will choose as partner-subsytem $(\mathbf{E}, \mathbf{B}, \alpha + \frac{\pi}{2})$ or $(\mathbf{E}, \mathbf{B}, \alpha - \frac{\pi}{2})$.

We may summarize this nonrelativistic approach as follows:

A real free field consists of two interacting subsystems (Σ_1, Σ_2) , and each subsystem is described by two partner-fields inside the $\alpha(x, y, z; t)$ -family

$$(\mathcal{E}, \mathcal{B}) = (\mathbf{E} \cos \alpha - \mathbf{B} \sin \alpha; \mathbf{E} \sin \alpha + \mathbf{B} \cos \alpha),$$

$\Sigma_1 = (\mathcal{E}_{\alpha_1}, \mathcal{B}_{\alpha_1}), \Sigma_2 = (\mathcal{E}_{\alpha_2}, \mathcal{B}_{\alpha_2})$, giving the same Maxwell stress-energy tensor, so the full stress energy tensor is the sum: $M(\Sigma_1, \Sigma_2) = \frac{1}{2}M(\Sigma_1) + \frac{1}{2}M(\Sigma_2)$. Each partner-field has electric and magnetic constituents, and each partner-field is determined by the other through $(\pm \frac{\pi}{2})$ - rotation-like transformation. Both partner-fields carry the

same energy-momentum : $M(\Sigma_1) = M(\Sigma_2)$, and the field propagates in space through minimizing the relation $I_1^2 + I_2^2 \geq 0$. The intrinsic dynamics of a free real time-dependent field establishes and maintains local energy-momentum exchange partnership between the two partner-fields, and since these partner-fields carry always the same stress-energy, the allowed exchange is necessarily simultaneous and in equal quantities, so, each partner-field conserves its energy-momentum during propagation.

5 Internal interaction and evolution in energy-momentum terms

In order to find how much is the locally exchanged energy-momentum we are going to interpret appropriately the above equation (**). Our object of interest, representing the integrity of a real time dependent electromagnetic field, is the couple $[(\mathbf{E}, \mathbf{B}); (-\mathbf{B}, \mathbf{E})]$ (the other case $[(\mathbf{E}, \mathbf{B}); (\mathbf{B}, -\mathbf{E})]$ is considered analogically). In view of the above considerations our equations should directly describe admissible energy-momentum exchange between these recognized two subsystems. Since these two subsystems are supposed to keep their recognizability during spacetime propagation, we have to take in view corresponding *admissible*, i.e. not leading to nonrecognizability but *real*, changes. It is supposed, of course, that these admissible real changes must be mathematically represented by tensor objects as it is, for example, how a vector field V changes along itself : if the change of V is ∇V , then the corresponding "projection" $\nabla_V V$ of V on ∇V measures the admissibility, and if $\nabla_V V = 0$ the change is admissible.

Let's denote such changes by $D(\mathbf{E}, \mathbf{B})$ and $D(-\mathbf{B}, \mathbf{E})$ for each partner-field. Their admissible nature now should formally mean that the values of the corresponding "projections"

$$\mathfrak{P} : D(\mathbf{E}, \mathbf{B}) \rightarrow (\mathbf{E}, \mathbf{B}) \quad \text{and} \quad \mathfrak{P} : D(-\mathbf{B}, \mathbf{E}) \rightarrow (-\mathbf{B}, \mathbf{E})$$

might be zero, i.e. not essential. Moreover, any available "cross-projections"

$$\mathfrak{P} : D(\mathbf{E}, \mathbf{B}) \rightarrow (-\mathbf{B}, \mathbf{E}) \quad \text{and} \quad \mathfrak{P} : D(-\mathbf{B}, \mathbf{E}) \rightarrow (\mathbf{E}, \mathbf{B})$$

must be appropriately equalizable, guaranteing in this way the required recognizability of each partner-field. From physical point of view the values of all these "projections" should have energy-momentum change nature in order to be considered as really available.

The left hand sides of the vacuum Maxwell equations (J.C.M.) we consider now as *non-zero admissible changes*, i.e. *their projections on the partner-fields might be zero or correspondingly equalizable*.

Following this assumption, the change object $D(\mathbf{E}, \mathbf{B})$ for the first partner-field (\mathbf{E}, \mathbf{B}) we naturally define as

$$D(\mathbf{E}, \mathbf{B}) := \left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}; \text{div} \mathbf{B} \right).$$

The corresponding "projection" $\mathfrak{P} : D(\mathbf{E}, \mathbf{B}) \rightarrow (\mathbf{E}, \mathbf{B})$

$$\mathfrak{P} [D(\mathbf{E}, \mathbf{B}); (\mathbf{E}, \mathbf{B})] = \mathfrak{P} \left[\left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}; \text{div} \mathbf{B} \right); (\mathbf{E}, \mathbf{B}) \right]$$

is suggested by the left hand side of the above energy-momentum exchange expressions (**), so we define it by :

$$\mathfrak{P} \left[\left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}; \text{div} \mathbf{B} \right); (\mathbf{E}, \mathbf{B}) \right] := \left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} + \mathbf{B} \text{div} \mathbf{B}.$$

For the second partner-field $(-\mathbf{B}, \mathbf{E})$, following the same procedure we obtain:

$$\begin{aligned} \mathfrak{P}[D(-\mathbf{B}, \mathbf{E}); (-\mathbf{B}, \mathbf{E})] &= \mathfrak{P}\left[\left(\text{rot}(-\mathbf{B}) + \frac{\partial \mathbf{E}}{\partial \xi}; \text{div} \mathbf{E}\right); (-\mathbf{B}, \mathbf{E})\right] \\ &= \left(\text{rot}(-\mathbf{B}) + \frac{\partial \mathbf{E}}{\partial \xi}\right) \times (-\mathbf{B}) + \mathbf{E} \text{div} \mathbf{E} = \left(\text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi}\right) \times \mathbf{B} + \mathbf{E} \text{div} \mathbf{E}. \end{aligned}$$

Hence, relation (**) takes the form

$$\mathfrak{P}[D(\mathbf{E}, \mathbf{B}); (\mathbf{E}, \mathbf{B})] = -\mathfrak{P}[D(-\mathbf{B}, \mathbf{E}); (-\mathbf{B}, \mathbf{E})].$$

We turn now to the mutual energy-momentum exchange between the two partner-fields $(\mathbf{E}, \mathbf{B}) \rightleftharpoons (-\mathbf{B}, \mathbf{E})$, keeping in mind that each of the two partner fields must keep its energy-momentum. The formal expressions are easy to obtain. In fact, in the case $D(-\mathbf{B}, \mathbf{E}) \rightarrow (\mathbf{E}, \mathbf{B})$ we have to "project" the change object for the second partner-field given by

$$D(-\mathbf{B}, \mathbf{E}) := \left(\text{rot}(-\mathbf{B}) + \frac{\partial \mathbf{E}}{\partial \xi}; \text{div} \mathbf{E}\right)$$

on the first partner-field (\mathbf{E}, \mathbf{B}) . We obtain:

$$\left(\text{rot}(-\mathbf{B}) + \frac{\partial \mathbf{E}}{\partial \xi}\right) \times \mathbf{E} + \mathbf{B} \text{div} \mathbf{E} = -\left(\text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi}\right) \times \mathbf{E} + \mathbf{B} \text{div} \mathbf{E}.$$

In the reverse case we have to "project" the change-object for the first partner-field (\mathbf{E}, \mathbf{B}) given by

$$D(\mathbf{E}, \mathbf{B}) := \left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}; \text{div} \mathbf{B}\right)$$

on the second partner-field $(-\mathbf{B}, \mathbf{E})$. We obtain

$$\left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}\right) \times (-\mathbf{B}) + \mathbf{E} \text{div} \mathbf{B} = -\left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}\right) \times \mathbf{B} + \mathbf{E} \text{div} \mathbf{B}.$$

We recall now the *special kind of dynamical equilibrium* between the two partner-fields, namely, *the two partner-fields necessarily carry always the same energy and momentum*:

$$M^{ij}(\mathbf{E}, \mathbf{B}, \alpha_1) = M^{ij}(\mathbf{E}, \mathbf{B}, \alpha_2), \quad \text{where} \quad |\alpha_1 - \alpha_2| = \frac{\pi}{2},$$

so, the mutual exchange is simultaneous and in equal quantities.

Hence, in view of the required recognizability of each of the two subsystems formally represented by the zero values of the two "self-projections", and the local dynamical equilibrium between these two subsystems formally represented by the simultaneous and the equal up to a sign mutual "projections" during any time period, the final equations read:

$$\begin{aligned} \left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}\right) \times \mathbf{E} + \mathbf{B} \text{div} \mathbf{B} &= 0, \\ \left(\text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi}\right) \times \mathbf{B} + \mathbf{E} \text{div} \mathbf{E} &= 0, \\ \left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}\right) \times \mathbf{B} - \mathbf{E} \text{div} \mathbf{B} + \left(\text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi}\right) \times \mathbf{E} - \mathbf{B} \text{div} \mathbf{E} &= 0. \end{aligned}$$

The third equation fixes, namely, that *the exchange of energy-momentum density between the two partner-fields is **simultaneous** and in **equal** quantities*, i.e. a **permanent dynamical equilibrium** between the two partner-fields holds.

Note that, this double-field viewpoint and the corresponding mutual energy-momentum exchange described by the last equation are essentially new moments.

The above equations can be given the following form in terms of differential forms. If g is the euclidean metric let's introduce the following notations:

$$\tilde{g}(\mathbf{E}) = \eta, \quad \tilde{g}(\mathbf{B}) = \beta, \quad \tilde{g}^{-1}(*\eta) = *\bar{\eta}, \quad \tilde{g}^{-1}(*\beta) = *\bar{\beta}.$$

Then we obtain

$$\begin{aligned} \tilde{g}(\text{rot} \mathbf{E} \times \mathbf{E}) &= i(\mathbf{E})\mathbf{d}\eta, & \tilde{g}(\text{rot} \mathbf{B} \times \mathbf{B}) &= i(\mathbf{B})\mathbf{d}\beta, \\ \tilde{g}(\text{rot} \mathbf{E} \times \mathbf{B}) &= i(\mathbf{B})\mathbf{d}\eta, & \tilde{g}(\text{rot} \mathbf{B} \times \mathbf{E}) &= i(\mathbf{E})\mathbf{d}\beta, \\ \mathbf{E} \text{div}(\mathbf{B}) &= i(*\bar{\eta})\mathbf{d} * \beta, & \mathbf{B} \text{div}(\mathbf{E}) &= i(*\bar{\beta})\mathbf{d} * \eta. \end{aligned}$$

Under these notations and relations the above three framed equations are respectively equivalent to:

$$\begin{aligned} i(\mathbf{E})\mathbf{d}\eta + i(*\bar{\beta})\mathbf{d} * \beta &= - * \left(\frac{\partial \beta}{\partial \xi} \wedge \eta \right) \\ i(\mathbf{B})\mathbf{d}\beta + i(*\bar{\eta})\mathbf{d} * \eta &= * \left(\frac{\partial \eta}{\partial \xi} \wedge \beta \right) \\ i(\mathbf{B})\mathbf{d}\eta - i(*\bar{\eta})\mathbf{d} * \beta + i(\mathbf{E})\mathbf{d}\beta - i(*\bar{\beta})\mathbf{d} * \eta &= * \left(\frac{\partial \eta}{\partial \xi} \wedge \eta - \frac{\partial \beta}{\partial \xi} \wedge \beta \right). \end{aligned}$$

Denoting now the Maxwell stress tensors of \mathbf{E} and \mathbf{B} correspondingly by $M(\mathbf{E})$ and $M(\mathbf{B})$, and introducing a new stress tensor $\mathbb{T} = M(\mathbf{E}) + M(\mathbf{B}) - M(\mathbf{E} + \mathbf{B})$ we obtain

$$\text{div} [M(\mathbf{E}) + M(\mathbf{B})] = 0, \quad \text{div} \mathbb{T} = \frac{\partial \mathbf{E}}{\partial \xi} \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial \xi} \times \mathbf{B}.$$

6 Some properties of the equations and their solutions

Consider the second equation and replace (\mathbf{E}, \mathbf{B}) according to

$$(\mathbf{E}, \mathbf{B}) \rightarrow (a\mathbf{E} - b\mathbf{B}, b\mathbf{E} + a\mathbf{B}),$$

where (a, b) are two constants. After the corresponding computation we obtain

$$\begin{aligned} a^2 \left[\left(\text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \times \mathbf{B} + \mathbf{E} \text{div} \mathbf{E} \right] &+ b^2 \left[\left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} + \mathbf{B} \text{div} \mathbf{B} \right] + \\ + ab \left[\left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{B} - \mathbf{E} \text{div} \mathbf{B} + \left(\text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \times \mathbf{E} - \mathbf{B} \text{div} \mathbf{E} \right] &= 0. \end{aligned}$$

Since the constants (a, b) are arbitrary the other two equations follow. The same property holds with respect to any of the three equations.

Corollary. The system of the three equations is invariant with respect to the transformation

$$(\mathbf{E}, \mathbf{B}) \rightarrow (a\mathbf{E} - b\mathbf{B}, b\mathbf{E} + a\mathbf{B}).$$

Writing down this transformation in the form

$$\begin{aligned} (\mathbf{E}, \mathbf{B}) &\rightarrow (\mathbf{E}', \mathbf{B}') = (\mathbf{E}, \mathbf{B}).\alpha(a, b) = (\mathbf{E}, \mathbf{B}) \begin{vmatrix} a & b \\ -b & a \end{vmatrix} \\ &= (a\mathbf{E} - b\mathbf{B}, b\mathbf{E} + a\mathbf{B}), \quad a = \text{const}, \quad b = \text{const}, \end{aligned}$$

we get a "right action" of the matrix α on the solutions. The new solution $(\mathbf{E}', \mathbf{B}')$ has energy and momentum densities equal to the old ones multiplied by $(a^2 + b^2)$. Hence, the space of all solutions factors over the action of the group of matrices of the kind

$$\alpha(a, b) = \begin{vmatrix} a & b \\ -b & a \end{vmatrix}, \quad (a^2 + b^2) \neq 0$$

in the sense, that the corresponding classes are determined by the value of $(a^2 + b^2)$.

All such matrices with nonzero determinant form a group with respect to the usual matrix product. A special property of this group is that it represents the symmetries of the canonical complex structure in \mathbb{R}^2 .

Clearly, all solutions to Maxwell pure field equations are solutions to our nonlinear equations, we shall call these solutions linear, and will not further be interested of them, we shall concentrate our attention on those solutions of our equations which satisfy the conditions

$$\text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \neq 0, \quad \text{rot } \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \neq 0, \quad \text{div } \mathbf{E} \neq 0, \quad \text{div } \mathbf{B} \neq 0.$$

These solutions we call further nonlinear.

We consider now some properties of the *nonlinear* solutions.

1. Among the nonlinear solutions there are no constant ones.
2. $\mathbf{E} \cdot \mathbf{B} = 0$; This is obvious, no proof is needed.
3. The following relations are also obvious:

$$\left(\text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \cdot \mathbf{B} = 0; \quad \left(\text{rot } \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \cdot \mathbf{E} = 0.$$

4. It is elementary to see from the last two relations that the classical Poynting energy-momentum balance equation follows.

5. $\mathbf{E}^2 = \mathbf{B}^2$.

In order to prove this let's take the scalar product of the first equation from the left by \mathbf{B} . We obtain

$$\mathbf{B} \cdot \left\{ \left(\text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} \right\} + \mathbf{B}^2 \text{div } \mathbf{B} = 0. \quad (1)$$

Now, multiplying the second equation from the left by \mathbf{E} and having in view $\mathbf{E} \cdot \mathbf{B} = 0$, we obtain

$$\mathbf{E} \cdot \left\{ \left(\text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{B} \right\} - \mathbf{E}^2 \text{div } \mathbf{B} = 0.$$

This last relation is equivalent to

$$-\mathbf{B} \cdot \left\{ \left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} \right\} - \mathbf{E}^2 \text{div} \mathbf{B} = 0. \quad (2)$$

Now, summing up (1) and (2), in view of $\text{div} \mathbf{B} \neq 0$, we come to the desired relation.

Properties **2.** and **5.** say that all nonlinear solutions are *null fields*, i.e. the two well known relativistic invariants $I_1 = \mathbf{B}^2 - \mathbf{E}^2$ and $I_2 = 2\mathbf{E} \cdot \mathbf{B}$ of the field are zero, and this property leads to optimisation of the inequality $I_1^2 + I_2^2 \geq 0$ (recall the eigen properties of Maxwell stress tensor), which in turn guarantees $\alpha(x, y, z; t)$ -invariance of $I_1 = I_2 = 0$.

6. The *helicity* property:

$$\mathbf{B} \cdot \left(\text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) - \mathbf{E} \cdot \left(\text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) = \mathbf{B} \cdot \text{rot} \mathbf{B} - \mathbf{E} \cdot \text{rot} \mathbf{E} = 0.$$

To prove this property we first multiply (vector product) the third equation from the right by \mathbf{E} , recall property **2.**, then multiply (scalar product) from the left by \mathbf{E} , recall again $\mathbf{E} \cdot \mathbf{B} = 0$, then multiply from the right (scalar product) by \mathbf{B} and recall property **5.**

Property **6.** suggests the following consideration. If \mathbf{V} is an arbitrary vector field on \mathbb{R}^3 then the quantity $\mathbf{V} \cdot \text{rot} \mathbf{V}$ is known as *local helicity* and its integral over the whole (compact) region occupied by \mathbf{V} is known as *integral helicity*, or just as *helicity* of \mathbf{V} . Hence, property **6.** says that the electric and magnetic constituents of a nonlinear solution generate the same helicities. If we consider (through the euclidean metric g) the 1-form $\tilde{g}(\mathbf{E})$ and denote by \mathbf{d} the exterior derivative on \mathbb{R}^3 , then

$$\tilde{g}(\mathbf{E}) \wedge \mathbf{d}\tilde{g}(\mathbf{E}) = \mathbf{E} \cdot \text{rot} \mathbf{E} \, dx \wedge dy \wedge dz,$$

so, the zero helicity says that the 1-form $\tilde{g}(\mathbf{E})$ defines a completely integrable Pfaff system: $\tilde{g}(\mathbf{E}) \wedge \mathbf{d}\tilde{g}(\mathbf{E}) = 0$. The nonzero helicity says that each of the 1-forms $\tilde{g}(\mathbf{E})$ and $\tilde{g}(\mathbf{B})$ defines non-integrable 1-dimensional Pfaff system, so the nonzero helicity defines corresponding curvature. Therefore the equality between the \mathbf{E} -helicity and the \mathbf{B} -helicity suggests to consider the corresponding integral helicity

$$\int_{\mathbb{R}^3} \tilde{g}(\mathbf{E}) \wedge \mathbf{d}\tilde{g}(\mathbf{E}) = \int_{\mathbb{R}^3} \tilde{g}(\mathbf{B}) \wedge \mathbf{d}\tilde{g}(\mathbf{B})$$

(when it takes finite nonzero values) as a measure of the spin properties of the solution.

We specially note that the equality of the local helicities defined by \mathbf{E} and \mathbf{B} holds also, as it is easily seen from the above relation, for the solutions of the linear Maxwell vacuum equations, but appropriate solutions giving well defined and time independent integral helicities in this case are missing. The next property shows that our nonlinear solutions admit such appropriate solutions giving finite constant integral helicities.

7. Example of nonlinear solution(s):

$$\begin{aligned} \mathbf{E} &= \left[\phi(x, y, \xi \pm z) \cos\left(-\kappa \frac{z}{\mathcal{L}_o} + \text{const}\right), \phi(x, y, \xi \pm z) \sin\left(-\kappa \frac{z}{\mathcal{L}_o} + \text{const}\right), 0 \right]; \\ \mathbf{B} &= \left[\pm \phi(x, y, \xi \pm z) \sin\left(-\kappa \frac{z}{\mathcal{L}_o} + \text{const}\right), \mp \phi(x, y, \xi \pm z) \cos\left(-\kappa \frac{z}{\mathcal{L}_o} + \text{const}\right), 0 \right], \end{aligned}$$

where $\phi(x, y, \xi \pm z)$ is an arbitrary positive function, $0 < \mathcal{L}_o < \infty$ is an arbitrary positive constant with physical dimension of length, and κ takes values ± 1 . Hence, we are allowed to choose the

function ϕ to have compact 3d-support, and since the energy density of this solution is $\phi^2 dx \wedge dy \wedge dz$ and the total energy is conserved according to property 4., these solutions will describe time-stable and space propagating with the speed of light finite field objects carrying finite integral energy.

Modifying now the corresponding helicity 3-forms to

$$\frac{2\pi\mathcal{L}_o^2}{c}\tilde{g}(\mathbf{E}) \wedge \mathbf{d}\tilde{g}(\mathbf{E}) = \frac{2\pi\mathcal{L}_o^2}{c}\tilde{g}(\mathbf{B}) \wedge \mathbf{d}\tilde{g}(\mathbf{B}),$$

then the corresponding 3d integral gives κTE , where $\kappa = \pm 1$, $T = 2\pi\mathcal{L}_o/c$ and $E = \int \phi^2 dx \wedge dy \wedge dz$ is the integral energy of the solution.

7 Scale factor

We consider the vector fields

$$\begin{aligned}\vec{\mathcal{F}} &= \text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} + \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} \text{div } \mathbf{B}, \\ \vec{\mathcal{M}} &= \text{rot } \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} - \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} \text{div } \mathbf{E},\end{aligned}$$

defined by a nonlinear solution.

It is obvious that on the solutions of Maxwell's vacuum equations $\vec{\mathcal{F}}$ and $\vec{\mathcal{M}}$ are equal to zero. Note also that under the transformation $(\mathbf{E}, \mathbf{B}) \rightarrow (-\mathbf{B}, \mathbf{E})$ we get $\vec{\mathcal{F}} \rightarrow -\vec{\mathcal{M}}$ and $\vec{\mathcal{M}} \rightarrow \vec{\mathcal{F}}$.

We shall consider now the relation between $\vec{\mathcal{F}}$ and \mathbf{E} , and between $\vec{\mathcal{M}}$ and \mathbf{B} on the nonlinear solutions of our equations assuming that $\vec{\mathcal{F}} \neq 0$ and $\vec{\mathcal{M}} \neq 0$.

Recalling $\mathbf{E} \cdot \mathbf{B} = 0$ we obtain

$$(\mathbf{E} \times \mathbf{B}) \times \mathbf{E} = -\mathbf{E} \times (\mathbf{E} \times \mathbf{B}) = -[\mathbf{E}(\mathbf{E} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{E} \cdot \mathbf{E})] = \mathbf{B}(\mathbf{E}^2),$$

and since $|\mathbf{E} \times \mathbf{B}| = |\mathbf{E}||\mathbf{B}|\sin(\mathbf{E}, \mathbf{B}) = \mathbf{E}^2 = \mathbf{B}^2$, we get

$$\vec{\mathcal{F}} \times \mathbf{E} = \left(\text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} + \mathbf{B} \text{div } \mathbf{B} = 0,$$

according to our first nonlinear equation.

In the same way, in accordance with our second nonlinear equation, we get $\vec{\mathcal{M}} \times \mathbf{B} = 0$. In other words, on the nonlinear solutions we obtain that $\vec{\mathcal{F}}$ is co-linear to \mathbf{E} and $\vec{\mathcal{M}}$ is co-linear to \mathbf{B} . Hence, we can write the relations

$$\vec{\mathcal{F}} = f_1 \cdot \mathbf{E}, \quad \vec{\mathcal{M}} = f_2 \cdot \mathbf{B},$$

where f_1 and f_2 are two functions, and of course, the interesting cases are $f_1 \neq 0, \infty$; $f_2 \neq 0, \infty$. Note that the physical dimension of f_1 and f_2 is the reciprocal to the dimension of coordinates, i.e. $[f_1] = [f_2] = [length]^{-1}$.

Note also that $\vec{\mathcal{F}}$ and $\vec{\mathcal{M}}$ are mutually orthogonal: $\vec{\mathcal{F}} \cdot \vec{\mathcal{M}} = 0$.

We shall prove now that $f_1 = f_2$. In fact, making use of the same formula for the double vector product, used above, we easily obtain

$$\begin{aligned}& \vec{\mathcal{F}} \times \mathbf{B} + \vec{\mathcal{M}} \times \mathbf{E} = \\ &= \left(\text{rot } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{B} + \left(\text{rot } \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \times \mathbf{E} - \mathbf{E} \text{div } \mathbf{B} - \mathbf{B} \text{div } \mathbf{E} = 0,\end{aligned}$$

in accordance with our third nonlinear equation. Therefore,

$$\begin{aligned}\vec{\mathcal{F}} \times \mathbf{B} + \vec{\mathcal{M}} \times \mathbf{E} &= \\ &= f_1 \mathbf{E} \times \mathbf{B} + f_2 \mathbf{B} \times \mathbf{E} = (f_1 - f_2) \mathbf{E} \times \mathbf{B} = 0.\end{aligned}$$

The assertion follows.

The relation $|\vec{\mathcal{F}}| = |\vec{\mathcal{M}}|$ is now obvious.

Note that the two relations $|\vec{\mathcal{F}}|^2 = |\vec{\mathcal{M}}|^2$ and $\vec{\mathcal{F}} \cdot \vec{\mathcal{M}} = 0$ and the duality correspondence $(\vec{\mathcal{F}}, \vec{\mathcal{M}}) \rightarrow (-\vec{\mathcal{M}}, \vec{\mathcal{F}})$ make us consider $\vec{\mathcal{F}}$ and $\vec{\mathcal{M}}$ as nonlinear analogs of \mathbf{E} and \mathbf{B} respectively.

These considerations suggest to introduce the quantity

$$\mathcal{L}(\mathbf{E}, \mathbf{B}) = \frac{1}{|f_1|} = \frac{1}{|f_2|} = \frac{|\mathbf{E}|}{|\vec{\mathcal{F}}|} = \frac{|\mathbf{B}|}{|\vec{\mathcal{M}}|},$$

which we call *scale factor*. Note that the physical dimension of \mathcal{L} is *length*. Hence, every nonlinear solution defines its own *scale factor* and, consequently, the nonlinear solutions factorize with respect to \mathcal{L} . It seems natural to connect the constant \mathcal{L}_o in the above given family of solutions with the so introduced scale factor. Assuming $\mathcal{L} = \mathcal{L}_o = \text{const}$, this could be done in the following way.

A careful look at the solutions above shows that at a given moment, e.g. $t = 0$, the finite spatial support of the function ϕ is built of continuous sheaf of nonintersecting helices along the coordinate z . Every such helix has a special length parameter $b = \lambda/2\pi$ giving the straight-line advance along the external straight-line axis (the coordinate z in our case) for a unit angle, and λ is the z -distance between two equivalent points on the same helix. So, we may put $\lambda = 2\pi\mathcal{L}_o = \text{const}$, hence, the z -size of the solution may, naturally, be bounded by $2\pi\mathcal{L}_o$.

Consider now a nonlinear solution with integral energy E and scale factor $\mathcal{L}_o = \text{const}$. Since this solution shall propagate in space with the speed of light c , we may introduce corresponding time period $T = 2\pi\mathcal{L}_o/c$, and define the quantity $\mathfrak{h} = E.T$, having physical dimension of "action". The temptation to separate a class of solutions requiring \mathfrak{h} to be equal to the Planck constant h is great, isn't it, especially if this T can be associated with some helix-like real periodicity during propagation?!

8 Conclusion

The basic idea we followed in this paper was to pass from linear to nonlinear equations in charge free classical electrodynamics through giving explicit physical sense of the equations as local stress-energy-momentum relations. This view on the subject naturally oriented our attention to the quantities carrying the necessary information - the corresponding stress-energy tensors. The existing knowledge about the structure and internal dynamics of free electromagnetic fields made us assume the notion for *two partner-fields internal structure*. These two partner-fields carry equal local energy densities, and realize local energy exchange without available interaction energy. Moreover, they strictly respect each other: the exchange is simultaneous and in equal quantities, so, both partner-fields keep their identity and recognizability. The corresponding internal dynamical structure appropriately unifies translation and rotation through unique space-time propagation with the fundamental velocity. All Maxwell solutions are duly respected. The new solutions admit finite spatial support, and these spatially finite solutions are of photon-like nature: they are time-stable, they demonstrate intrinsically compatible translational-rotational dynamical structure, they carry finite energy and intrinsically determined integral characteristic \mathfrak{h} of action nature through appropriate scale factor $\mathcal{L}_o = \text{const}$. Their integral energy E satisfies relation of the form identical to the Planck formula $E.T = \mathfrak{h}$.

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